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Error analysis for a sinh transformation used in evaluating nearly singular boundary element integrals

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Abstract

In the two-dimensional boundary element method, one often needs to evaluate numerically integrals of the form $\int_{-1}^{1} g(x)\, j(x)\, f((x-a)^2+b^2)\,\mathrm{d}x$ where j^2 is a quadratic, g is a polynomial and f is a rational, logarithmic or algebraic function with a singularity at zero. The constants a and b are such that $-1 \leqslant a \leqslant 1$ and $0 < b \ll 1$ so that the singularities of f will be close to the interval of integration. In this case the direct application of Gauss–Legendre quadrature can give large truncation errors. By making the transformation $x = a + b\sinh(\mu u - \eta)$, where the constants μ and η are chosen so that the interval of integration is again $[-1, 1]$, it is found that the truncation errors arising, when the same Gauss–Legendre quadrature is applied to the transformed integral, are much reduced. The asymptotic error analysis for Gauss–Legendre quadrature, as given by Donaldson and Elliott [A unified approach to quadrature rules with asymptotic estimates of their remainders, SIAM J. Numer. Anal. 9 (1972) 573–602], is then used to explain this phenomenon and justify the transformation.

Keywords: Numerical integration; Boundary element method; Nearly singular integrals; Nonlinear coordinate transformation; Sinh function

1. Introduction

The boundary element method in two dimensions requires the numerical evaluation of many line integrals. For integrals which are non-singular (the source point is well removed from the element over which the integration is required), a straightforward application of Gaussian quadrature is sufficient to obtain accurate numerical values for these integrals; see, for example, Brebbia and Dominguez [2]. For integrals which are singular (the source point being on the element), several transformation methods have been devised to improve the accuracy of the numerical evaluation of the integrals. These transformation methods use either standard Gaussian quadrature points which are clustered near the singular point by means of a polynomial transformation [9,13–19], or split the interval of integration at the singular point, and then cluster the integration points towards the end-points of the intervals [6–8,10].

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A class of integrals which lies between the two types of integrals mentioned above is that of "nearly singular" integrals where the source point is close to the interval of integration, but not on it. To be more explicit we shall consider for $k = 1, 2$, and 3 the integrals $I_k(g; j)$ where

$$I_k(g; j) := \int_{-1}^{1} g(x)j(x)h_k(x)\,\mathrm{d}x. \tag{1.1}$$

The functions h_k will be defined for $z \in \mathbb{C}$ as

$$h_1(z) := \tfrac{1}{2}\log((z-a)^2 + b^2) = \tfrac{1}{2}\log((z-z_0)(z-\overline{z_0})), \tag{1.2}$$

$$h_2(z) := \frac{1}{(z-a)^2 + b^2} = \frac{1}{(z-z_0)(z-\overline{z_0})}, \tag{1.3}$$

$$h_3(z) := ((z-a)^2 + b^2)^{\lambda} = ((z-z_0)(z-\overline{z_0}))^{\lambda}, \tag{1.4}$$

where λ is not an integer. The point z_0 is defined to be

$$z_0 := a + \mathrm{i}b, \tag{1.5}$$

where $a \in \mathbb{R}$ and, without loss of generality, we shall henceforth assume that $b > 0$. Thus, the functions h_k have either branch points or poles at the points $a \pm \mathrm{i}b$. We shall assume that these points are "close" to the basic interval $[-1, 1]$ and, in Section 2, we shall define what we mean by "close".

Returning to (1.1), we shall assume that g is a real polynomial which does not have zeros at z_0, $\overline{z_0}$. Finally, the function j will be considered to have arisen from the Jacobian of the transformation, from the original, possibly curved, element over which the integral was taken, on to the interval $[-1, 1]$. We shall assume either that j is constant or of the form

$$j(z) := \sqrt{(z-c)^2 + d^2} = ((z-z_1)(z-\overline{z_1}))^{1/2}, \tag{1.6}$$

where $z_1 = c + \mathrm{i}d$. Furthermore, we assume that $z_1 \neq z_0$ and that z_1 is "further away" from the interval $[-1, 1]$ than is z_0, (see Section 2).

In Johnston and Elliott [11] we have considered some numerical examples of these integrals. Firstly, we have used 10 and 20 point Gauss–Legendre quadrature to evaluate numerically some specific integrals $I_k(g; j)$ with $k = 1, 2$ and 3. Then we have made a transformation of the variable of integration by writing

$$x = a + b\sinh(\mu u - \eta), \tag{1.7}$$

where μ and η are chosen so that the interval $-1 \leqslant x \leqslant 1$ is mapped onto the interval $-1 \leqslant u \leqslant 1$ with $x = \pm 1$ corresponding to $u = \pm 1$, respectively. We have then applied 10 and 20 point Gauss–Legendre quadrature to the transformed integrals. From the examples considered in Johnston and Elliott [11] we have found that, at all times, the use of transformation (1.7) can lead to a dramatic reduction in the truncation errors. In this paper we shall make use of the asymptotic error analysis for Gauss–Legendre quadrature, as given by Donaldson and Elliott [3], in order to estimate the errors for both the transformed and untransformed integrals. As we shall see in Section 6, this analysis gives excellent estimates for the truncation errors in both cases and, from the analytic forms of the remainders, we can see why the transformed integrals have considerably smaller truncation errors. To this end we need to consider the family of confocal ellipses with foci at the points $(-1, 0)$ and $(1, 0)$. It turns out that the singularities in the transformed integrals lie on "larger" ellipses than do the singularities of the untransformed integrals. For large n, this has a profound effect on the size of the truncation errors.

In the next section, we shall consider the geometry of the family of confocal ellipses with foci at $(\pm 1, 0)$. In Section 3 we shall consider what happens to the point (a, b) under the transformation (1.7). In Section 4 we shall obtain asymptotic estimates for the truncation errors when Gauss–Legendre quadrature is applied to the untransformed integrals. In Section 5 we shall consider corresponding estimates for the transformed integrals. A comparison of the asymptotic estimates of the truncation errors with the computed truncation errors will be given in Section 6, for four examples.

2. A family of confocal ellipses

It is well known, see, for example, Kitchen [12, Chapter 12], that if $\alpha > 1$ then

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\alpha^2 - 1} = 1 \tag{2.1}$$

is the equation of an ellipse in the (x, y) plane with foci at $(\pm 1, 0)$ and semi-axes given by α and $\sqrt{\alpha^2 - 1} > 0$.

Consider now the complex z-plane, where $z = x + \mathrm{i}y$, and points such that

$$\left| z + \sqrt{z^2 - 1} \right| = \rho > 1. \tag{2.2}$$

From (2.2) we have

$$z + \sqrt{z^2 - 1} = \rho \mathrm{e}^{\mathrm{i}\phi} \tag{2.3}$$

say, where $0 \leqslant \phi \leqslant 2\pi$. Then, trivially,

$$z - \sqrt{z^2 - 1} = \frac{1}{\rho} \mathrm{e}^{-\mathrm{i}\phi}, \tag{2.4}$$

so that on adding (2.3) and (2.4) we find

$$z = x + \mathrm{i}y = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right)\cos\phi + \frac{\mathrm{i}}{2}\left(\rho - \frac{1}{\rho}\right)\sin\phi. \tag{2.5}$$

On equating real and imaginary parts in (2.5) and eliminating ϕ we obtain

$$\frac{x^2}{(\frac{1}{2}(\rho + 1/\rho))^2} + \frac{y^2}{(\frac{1}{2}(\rho - 1/\rho))^2} = 1. \tag{2.6}$$

On comparing (2.1) with (2.6) we see that points satisfying (2.2) lie on an ellipse with semi-major axis $\alpha = \frac{1}{2}(\rho + 1/\rho) > 1$, semi-minor axis given by $\frac{1}{2}(\rho - 1/\rho) = \sqrt{\alpha^2 - 1} > 0$ and with foci at $(\pm 1, 0)$.

Let $\mathscr{C}_\rho$ denote the ellipse given by (2.2) and throughout we shall assume that $\mathscr{C}_\rho$ is described in the positive (i.e., anticlockwise) direction. As we vary $\rho > 1$, we obtain a family of confocal ellipses.

The integrands of $I_k(g; j)$ all have singularities in the complex plane at the points $z_0, \overline{z_0}$ where $z_0 = a + \mathrm{i}b$ for $a \in \mathbb{R}$ and $b > 0$. Having fixed z_0, we wish to determine on which ellipse $\mathscr{C}_\rho$ this point lies. It is well known that if $P(a, b)$ lies on the ellipse (2.1) with semi-major axis α then the sum of the distances from P to the foci is equal to 2α. (Kitchen calls this the "string property" of the ellipse, see [12, p. 534].) Since the foci are at the points $(\pm 1, 0)$ we have at once that $\alpha = \alpha(a, b)$ is given by

$$\alpha(a, b) = \tfrac{1}{2}\left\{\sqrt{(a + 1)^2 + b^2} + \sqrt{(a - 1)^2 + b^2}\right\}. \tag{2.7}$$

Thus given the point (a, b), Eq. (2.7) gives α from which $\rho = \rho(a, b)$ is given by

$$\rho(a, b) = \alpha(a, b) + \sqrt{\alpha^2(a, b) - 1}. \tag{2.8}$$

Suppose (a_1, b_1) with $b_1 > 0$ is another point in the plane. We shall have one of the following:

$$\begin{cases} \dfrac{a_1^2}{\alpha^2} + \dfrac{b_1^2}{\alpha^2 - 1} = 1, \\ \dfrac{a_1^2}{\alpha^2} + \dfrac{b_1^2}{\alpha^2 - 1} > 1, \\ \dfrac{a_1^2}{\alpha^2} + \dfrac{b_1^2}{\alpha^2 - 1} < 1. \end{cases} \tag{2.9}$$

In the first case, the points (a, b) and (a_1, b_1) lie on the same ellipse $\mathscr{C}_\rho$ and we shall say that they are at the "same distance" from the interval $[-1, 1]$ which, incidentally, corresponds to the degenerate ellipse with $\rho = 1$. In the second case, the point (a_1, b_1) lies *outside* the ellipse $\mathscr{C}_\rho$ and will lie on the ellipse $\mathscr{C}_{\rho_1}$ say, with $\rho_1 > \rho$. We shall say the point (a_1, b_1) is "further away" from $[-1, 1]$ than is the point (a, b). In the third case, the point (a_1, b_1) lies *inside* the ellipse $\mathscr{C}_\rho$ and we shall say that (a_1, b_1) is "closer" to $[-1, 1]$ than is the point (a, b).

Some elementary properties of the function α are given in the following lemma.

Lemma 2.1. (i) *For all* $a \in \mathbb{R}$ and $b > 0$

$$\alpha(-a, b) = \alpha(a, b).$$

(ii) *For a given* $b > 0$ *and* $a \geqslant 0$, $\alpha(\cdot, b)$ *is a monotonic increasing function with*

$$\alpha(a, b) \geqslant \alpha(0, b) = \sqrt{1 + b^2}. \tag{2.10}$$

Proof. (i) This follows immediately from (2.7).

(ii) From (2.7) we have at once that $\alpha(0, b) = \sqrt{1 + b^2}$. Again, from (2.7),

$$\frac{\partial \alpha}{\partial a} = \frac{1}{2}\left\{\frac{a+1}{\sqrt{(a+1)^2 + b^2}} + \frac{a-1}{\sqrt{(a-1)^2 + b^2}}\right\}. \tag{2.11}$$

If $a \geqslant 1$, we see that both terms are non-negative so that $\partial\alpha/\partial a \geqslant 0$ and $\alpha(a, b)$ is monotonic increasing for $a \geqslant 1$.

For $0 \leqslant a \leqslant 1$ we have from (2.11) that

$$\begin{aligned}\frac{\partial \alpha}{\partial a} &= \frac{(1+a)\sqrt{(a-1)^2 + b^2} - (1-a)\sqrt{(a+1)^2 + b^2}}{2\sqrt{(a+1)^2 + b^2}\sqrt{(a-1)^2 + b^2}} \\ &= \frac{4ab^2}{2\sqrt{(a+1)^2 + b^2}\sqrt{(a-1)^2 + b^2}}\,\frac{1}{\left\{(1+a)\sqrt{(a-1)^2 + b^2} + (1-a)\sqrt{(a+1)^2 + b^2}\right\}}.\end{aligned} \tag{2.12}$$

From (2.12) we see that $\partial\alpha/\partial a \geqslant 0$ for $0 \leqslant a \leqslant 1$ so that $\alpha(a, b)$ is also monotonic increasing for $0 \leqslant a \leqslant 1$. This proves (ii). □

We shall now consider the transformation (1.7).

3. The sinh Transformation

For the integrals $I_k(g; j)$, see (1.1), we make a transformation of the variable of integration by writing

$$x = a + b\sinh(\mu(a, b)u - \eta(a, b)), \tag{3.1}$$

where μ and η are chosen so that the interval of integration $-1 \leqslant x \leqslant 1$ corresponds to $-1 \leqslant u \leqslant 1$ with $u = \pm 1$ corresponding to $x = \pm 1$, respectively. This gives

$$\mu(a, b) := \frac{1}{2}\left\{\operatorname{arcsinh}\left(\frac{1+a}{b}\right) + \operatorname{arcsinh}\left(\frac{1-a}{b}\right)\right\} \tag{3.2}$$

and

$$\eta(a, b) := \frac{1}{2}\left\{\operatorname{arcsinh}\left(\frac{1+a}{b}\right) - \operatorname{arcsinh}\left(\frac{1-a}{b}\right)\right\}. \tag{3.3}$$

Note that although μ and η depend upon a and b, we shall at times suppress these arguments when it is not necessary to emphasise this dependence.

With the transformation (3.1) we find from (1.1)–(1.4) that

$$I_k(g;\, j) = \int_{-1}^{1} G(u)J(u)H_k(u)\,\mathrm{d}u \tag{3.4}$$

say, where we define

$$G(u) := \mu g(a + b\sinh(\mu u - \eta)), \tag{3.5}$$

$$J(u) := j(a + b\sinh(\mu u - \eta)), \tag{3.6}$$

$$H_1(u) := b\cosh(\mu u - \eta)\log(b\cosh(\mu u - \eta)), \tag{3.7}$$

$$H_2(u) := 1/(b\cosh(\mu u - \eta)), \tag{3.8}$$

$$H_3(u) := (b\cosh(\mu u - \eta))^{2\lambda+1}. \tag{3.9}$$

As shown by examples in Johnston and Elliott [11], the truncation errors arising when n-point Gauss–Legendre quadrature is applied to $I_k(g;\, j)$ as defined by (3.4)–(3.9) are considerably less than the truncation errors arising when the same quadrature rule is applied to $I_k(g;\, j)$ as defined by (1.1) et seq. In this paper we shall obtain asymptotic estimates, assuming n is large, for these truncation errors so that we can compare them analytically. In order to do this we shall use the complex variable techniques as given in Donaldson and Elliott [3]. As noted in [3], although we assume that n is "large", very often the asymptotic error estimates are good (i.e., giving one correct significant figure) even for "modest" values of n. To that end we rewrite (3.1) as

$$z = a + b\sinh(\mu(a, b)w - \eta(a, b)), \tag{3.10}$$

where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$. The untransformed integrals, (1.1) et seq are such that the integrand has singularities at the points $a \pm \mathrm{i}b$ in the z-plane. For the transformed integrals (3.4) et seq the integrands have singularities in the complex w-plane at points where

$$\cosh(\mu(a, b)w - \eta(a, b)) = 0, \tag{3.11}$$

or at the points w_k, $k \in \mathbb{Z}$, where

$$w_k = \frac{\eta(a, b)}{\mu(a, b)} + \mathrm{i}\frac{(k + \frac{1}{2})\pi}{\mu(a, b)}. \tag{3.12}$$

In particular, we shall consider the points $s \pm \mathrm{i}t$, say, which are closest to the interval $[-1, 1]$. That is, we have

$$s(a, b) \pm \mathrm{i}t(a, b) = \frac{\eta(a, b)}{\mu(a, b)} \pm \mathrm{i}\frac{\pi}{2\mu(a, b)}. \tag{3.13}$$

If, in the z-plane, the points $a \pm \mathrm{i}b$ lie on the ellipse $\mathscr{C}_\rho$ and if, in the w-plane, the points $s \pm \mathrm{i}t$ lie on the ellipse $\mathscr{C}_{\rho_1}$, we wish to compare ρ_1 with ρ in order to see which of the points is furthest away from the basic interval $[-1, 1]$. To do this we need some preliminary results.

Lemma 3.1. *Suppose $b > 0$. Then*

(i) *$\mu(\cdot, b)$ is a continuous, positive and even function on $\mathbb{R}$;*
(ii) *$\mu(\cdot, b)$ is monotonic decreasing on $\mathbb{R}^+$;*
(iii)

$$\mu(a, b) \leqslant \begin{cases} 1/b & \text{if } 0 \leqslant a \leqslant 1, \\ 1/\sqrt{b^2 + (a-1)^2} & \text{if } a \geqslant 1. \end{cases} \tag{3.14}$$

Proof. (i) Since the arcsinh function is continuous on $\mathbb{R}$, it follows from (3.2) that, for a fixed $b > 0$, $\mu(\cdot, b)$ is continuous on $\mathbb{R}$. From (3.2) it follows at once that $\mu(-a, b) = \mu(a, b)$ so that $\mu(\cdot, b)$ is an even function. From the definition of the arcsinh function as an integral, see, for example, Abramowitz and Stegun [1, Section 4.6.1] and recalling that it is an odd function we have from (3.2) that

$$\begin{aligned}\mu(a,b) &= \frac{1}{2}\left\{\operatorname{arcsinh}\left(\frac{a+1}{b}\right) - \operatorname{arcsinh}\left(\frac{a-1}{b}\right)\right\} \\ &= \frac{1}{2}\int_{(a-1)/b}^{(a+1)/b} \frac{\mathrm{d}t}{\sqrt{1+t^2}} \\ &= \frac{1}{2}\int_{a-1}^{a+1} \frac{\mathrm{d}\xi}{\sqrt{b^2+\xi^2}},\end{aligned} \tag{3.15}$$

on writing $t = \xi/b$. Since the integrand is positive and since $a-1 < a+1$, it follows that $\mu(\cdot, b)$ is positive.

(ii) From (3.15) we have

$$\frac{\partial\mu(a,b)}{\partial a} = \frac{1}{2}\left\{\frac{1}{\sqrt{b^2+(a+1)^2}} - \frac{1}{\sqrt{b^2+(a-1)^2}}\right\} \tag{3.16}$$

and this is negative for all $a > 0$. Hence $\mu(\cdot, b)$ is monotonic decreasing on $\mathbb{R}^+$.

(iii) If $0 \leqslant a \leqslant 1$ we see that the interval of integration in (3.15) includes the origin at which the integrand takes its maximum value of $1/b$. Since the interval of integration is of length 2, independent of a, it follows at once that $\mu(a,b) \leqslant 1/b$, for $0 \leqslant a \leqslant 1$.

If $a \geqslant 1$ the integrand in (3.15) takes its maximum value of $1/\sqrt{b^2+(a-1)^2}$, at the point $(a-1)$. The second inequality in (3.14) then follows at once. □

We have a similar set of results for the function $\eta(\cdot, b)$.

Lemma 3.2. *Suppose* $b > 0$. *Then*

(i) $\eta(\cdot, b)$ *is a continuous*, *odd and monotonic increasing function on* $\mathbb{R}$;
(ii) $\eta(\cdot, b)$ *is positive on* $\mathbb{R}^+$.

Proof. (i) Since $b > 0$ and the arcsinh function is continuous on $\mathbb{R}$, the continuity of $\eta(\cdot, b)$ on $\mathbb{R}$ follows from (3.3). The fact that $\eta(-a, b) = -\eta(a, b)$ for all $a \in \mathbb{R}$ again follows from (3.3). Finally from (3.3) we have that

$$\frac{\partial\eta(a,b)}{\partial a} = \frac{1}{2}\left\{\frac{1}{\sqrt{b^2+(1+a)^2}} + \frac{1}{\sqrt{b^2+(1-a)^2}}\right\}, \tag{3.17}$$

which is positive for all $a \in \mathbb{R}$ so that $\eta(\cdot, b)$ is monotonic increasing on $\mathbb{R}$.

(ii) This follows immediately from (i). Since $\eta(\cdot, b)$ is odd then $\eta(0, b) = 0$ and since it is monotonic increasing it must be positive on $\mathbb{R}^+$. □

There is one further inequality which is worth noting.

Lemma 3.3. *Suppose* $b > 0$. *Then*

$$\eta(a,b) \geqslant a\mu(a,b) \quad \textit{if } a \geqslant 1 \tag{3.18}$$

and

$$\eta(a,b) \leqslant a\mu(a,b) \quad \textit{if } 0 \leqslant a \leqslant 1. \tag{3.19}$$

Proof. Suppose first that $a \geqslant 1$ and define

$$F(a,b) := \eta(a,b) - a\mu(a,b). \tag{3.20}$$

From (3.2) and (3.3) we have at once that $F(1, b) = 0$. Now

$$\frac{\partial F(a,b)}{\partial a} = \frac{\partial \eta(a,b)}{\partial a} - a\frac{\partial \mu(a,b)}{\partial a} - \mu(a,b) \tag{3.21}$$

so that from (3.16) and (3.17) we find

$$\frac{\partial F(a,b)}{\partial a} = \frac{a+1}{2}\frac{1}{\sqrt{b^2+(a-1)^2}} - \frac{a-1}{2}\frac{1}{\sqrt{b^2+(a+1)^2}} - \mu(a,b). \tag{3.22}$$

From (3.14) and (3.22) we have that for $a \geqslant 1$

$$\frac{\partial F(a,b)}{\partial a} \geqslant \frac{a-1}{2}\left\{\frac{1}{\sqrt{b^2+(a-1)^2}} - \frac{1}{\sqrt{b^2+(a+1)^2}}\right\} \geqslant 0. \tag{3.23}$$

Hence for a given $b > 0$, the function $F(\cdot, b)$ is monotonic increasing on $[1, \infty)$ and since $F(1, b) = 0$ we have $F(a, b) \geqslant 0$ for all $a \geqslant 1$. This proves (3.18).

Suppose now that $0 < a \leqslant 1$. From (3.2) and (3.3) we have

$$\mu(1/a, b/a) = \eta(a, b)$$

and

$$\eta(1/a, b/a) = \mu(a, b). \tag{3.24}$$

Since $1/a \geqslant 1$ and $b/a > 0$ we have by (3.18) that

$$\eta(1/a, b/a) \geqslant (1/a)\mu(1/a, b/a). \tag{3.25}$$

From (3.24) again we have $a\mu(a, b) \geqslant \eta(a, b)$ which is (3.19) for $0 < a \leqslant 1$. Since, from (3.3), (3.19) is trivially true when $a = 0$, the lemma is proved. □

We are now in a position to compare the proximity of the points (a, b) and (s, t) to the basic interval $[-1, 1]$. Because of the symmetry of the confocal ellipses we shall assume (a, b) is such that $a \geqslant 0$ and $b > 0$.

Theorem 3.4. *Suppose $a \geqslant 0$ and $b > 0$ are given. The point (a, b) in the z-plane is closer to the basic interval $[-1, 1]$ than is the point (s, t), as defined in (3.13), in the w-plane.*

Proof. From (2.7) we have that the point (a, b) lies on an ellipse with semi-major axis $\alpha(a, b)$ whereas (s, t) lies on an ellipse with semi-major axis $\alpha(s, t)$. The theorem is proved if we can show that $\alpha(s, t) > \alpha(a, b)$.

First, suppose that $a \geqslant 1$. From the definition of s as given in (3.13) and from (3.18) we have that $s \geqslant a$. It follows from (2.7) and Lemma 2.1(ii) that

$$\alpha(s,t) \geqslant \tfrac{1}{2}\left\{\sqrt{(a+1)^2+t^2(a,b)} + \sqrt{(a-1)^2+t^2(a,b)}\right\}, \tag{3.26}$$

and the theorem will be proved if $t(a, b) > b$. From (3.13), we need to show that

$$b\mu(a,b) < \frac{\pi}{2}. \tag{3.27}$$

But from Lemma 3.1(iii) we have that

$$b\mu(a,b) \leqslant \frac{b}{\sqrt{b^2+(a-1)^2}} \leqslant 1 < \frac{\pi}{2}. \tag{3.28}$$

Thus, for $a \geqslant 1$ and $b > 0$ we have that the point (s, t) is further away from the interval $[-1, 1]$ than is (a, b).

Suppose now that $0 \leqslant a \leqslant 1$ with $b > 0$. From Lemma 2.1(ii) and (2.7) we have that $\alpha(s(a,b), t(a,b)) \geqslant \alpha(0, t(a,b)) = \sqrt{1+t^2(a,b)}$. Again, since $0 \leqslant a \leqslant 1$, we have that $\alpha(a,b) \leqslant \alpha(1,b) = (b+\sqrt{4+b^2})/2$. If for $0 \leqslant a \leqslant 1$ and $b > 0$ we can show that $\sqrt{1+t^2(a,b)} > (b+\sqrt{4+b^2})/2$ then we shall have the required result since

$$\alpha(s(a,b), t(a,b)) \geqslant \alpha(0, t(a,b)) = \sqrt{1+t^2(a,b)} > \left(b+\sqrt{4+b^2}\right)/2 = \alpha(1,b) \geqslant \alpha(a,b). \tag{3.29}$$

Now from (3.13), $t(a,b) = \pi/(2\mu(a,b))$ and since, from Lemma 3.1(ii) $\mu(\cdot, b)$ is monotonic decreasing on $[0, 1]$ it follows that $t(a,b) \geqslant \pi/(2\mu(0,b)) = \pi/(2\operatorname{arcsinh}(1/b))$, from (3.2). From (3.29) we need to show that

$$\sqrt{1+\frac{\pi^2}{4\operatorname{arcsinh}^2(1/b)}} > \frac{b+\sqrt{4+b^2}}{2}, \tag{3.30}$$

for all $b > 0$. On squaring (3.30) and then taking the square root we need to show that

$$\sqrt{2}\sqrt{b}\sqrt{b+\sqrt{4+b^2}}\operatorname{arcsinh}(1/b) < \pi \tag{3.31}$$

for all $b > 0$. On writing $b = 1/x$, (3.31) is equivalent to showing that

$$\frac{\sqrt{2}\sqrt{1+\sqrt{1+4x^2}}}{x}\operatorname{arcsinh} x < \pi \tag{3.32}$$

for all $x > 0$. To prove this inequality let us consider $0 < x \leqslant 1$ and $x \geqslant 1$ separately.

On the interval $0 < x \leqslant 1$ let us write

$$\begin{aligned} f_1(x) &:= \sqrt{2}\sqrt{1+\sqrt{1+4x^2}}, \\ f_2(x) &:= (\operatorname{arcsinh} x)/x. \end{aligned} \tag{3.33}$$

If we define $f_2(0)$ by $\lim_{x\to 0^+} f_2(x)$ then $f_2(0) = 1$. Now for $0 \leqslant x \leqslant 1$ we have $\max_{0\leqslant x\leqslant 1} f_1(x) = \sqrt{2}\sqrt{1+\sqrt{5}}$ and $\max_{0\leqslant x\leqslant 1} f_2(x) = 1$ so that from (3.32) we have

$$\sqrt{2}\sqrt{1+\sqrt{1+4x^2}}\operatorname{arcsinh}(x)/x \leqslant \sqrt{2}\sqrt{1+\sqrt{5}} < \pi. \tag{3.34}$$

On the interval $1 \leqslant x < \infty$, let us write

$$\begin{aligned} f_3(x) &:= \sqrt{2}\sqrt{1+\sqrt{1+4x^2}}/\sqrt{x}, \\ f_4(x) &:= (\operatorname{arcsinh} x)/\sqrt{x}. \end{aligned} \tag{3.35}$$

Since $x \geqslant 1$, it readily follows that $f_3(x) \leqslant f_3(1) = \sqrt{2}\sqrt{1+\sqrt{5}}$. For $x \geqslant 1$, since $x+\sqrt{1+x^2} \leqslant (\sqrt{2}+1)x$ it follows that $f_4(x) = \log(x+\sqrt{1+x^2})/\sqrt{x} \leqslant f_5(x)$ say, where

$$f_5(x) := \log((\sqrt{2}+1)x)/\sqrt{x}. \tag{3.36}$$

Elementary calculus shows that f_5 has a maximum at the point x_0, say where

$$x_0 = \mathrm{e}^2/(\sqrt{2}+1). \tag{3.37}$$

From (3.35)–(3.37) we have, for $1 \leqslant x < \infty$, that

$$f_4(x) \leqslant f_5(x_0) = 2(1+\sqrt{2})/e, \tag{3.38}$$

Table 1
Values of $\rho(0, t(0, b))/\rho(0, b)$

b	$\rho(0, t(0, b))/\rho(0, b)$
1	1.5847
0.1	1.4958
0.01	1.3262
0.001	1.2266
0.0001	1.1710

so that

$$\frac{\sqrt{2}\sqrt{1+\sqrt{1+4x^2}}}{x}\operatorname{arcsinh} x \leqslant \frac{2^{3/2}\sqrt{1+\sqrt{2}}\sqrt{1+\sqrt{5}}}{e} < 2.91 < \pi, \tag{3.39}$$

as required.

This establishes the inequality (3.31) for all $b > 0$ and the theorem is proved. □

Theorem 3.4 is as much as we have been able to prove regarding the relative positions of the points (a, b) and $(s(a, b), t(a, b))$. In the light of the truncation error estimates to be discussed in the next section, it would useful to have an estimate for $\rho(s(a, b), t(a, b))/\rho(a, b)$, ρ being defined by Eqs. (2.7) and (2.8). After some numerical experiments we are led to conjecture that

$$\begin{aligned}\frac{\rho(s(a,b),t(a,b))}{\rho(a,b)} &\geqslant \frac{\rho(0,t(0,b))}{\rho(0,b)} \\ &= \frac{\pi/2+\sqrt{\pi^2/4+\operatorname{arcsinh}^2(1/b)}}{\left(b+\sqrt{1+b^2}\right)\operatorname{arcsinh}(1/b)},\end{aligned} \tag{3.40}$$

for all $a \in \mathbb{R}$ and $b > 0$. In Table 1 we have considered this lower bound for various values of b.

The significance of these results will become apparent in the next three sections when we consider the truncation errors which arise when n-point Gauss–Legendre quadrature is applied to both the original and the transformed integrals.

4. Truncation error estimates for the untransformed integrals

In order to obtain estimates for the truncation errors we apply the analysis given by Donaldson and Elliott [3]. This may be summarised as follows. To evaluate $\int_{-1}^{1} f(x)\,\mathrm{d}x$ we write

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = Q_n f + E_n f, \tag{4.1}$$

where $Q_n f$ denotes the n-point Gauss–Legendre quadrature sum and $E_n f$ is the truncation error which we shall attempt to estimate for $n \geqslant 1$. If the definition of the function f can be continued from the interval $[-1, 1]$ into the complex plane then we have

$$E_n f = \frac{1}{2\pi\mathrm{i}}\int_{\mathscr{C}_\rho} k_n(z) f(z)\,\mathrm{d}z. \tag{4.2}$$

As before, the contour $\mathscr{C}_\rho$ with $\rho > 1$ denotes a member of the family of confocal ellipses with foci at $(-1, 0)$ and $(1, 0)$ described in the positive (i.e., anticlockwise) direction with ρ being chosen so that f is analytic on and within $\mathscr{C}_\rho$. The function k_n is independent of f and depends only upon the fact that we are using n-point Gauss–Legendre quadrature. It is analytic in $\mathbb{C}\backslash[-1, 1]$.

From Donaldson and Elliott [3] we have that

$$k_n(z) = \frac{2Q_n(z)}{P_n(z)}, \tag{4.3}$$

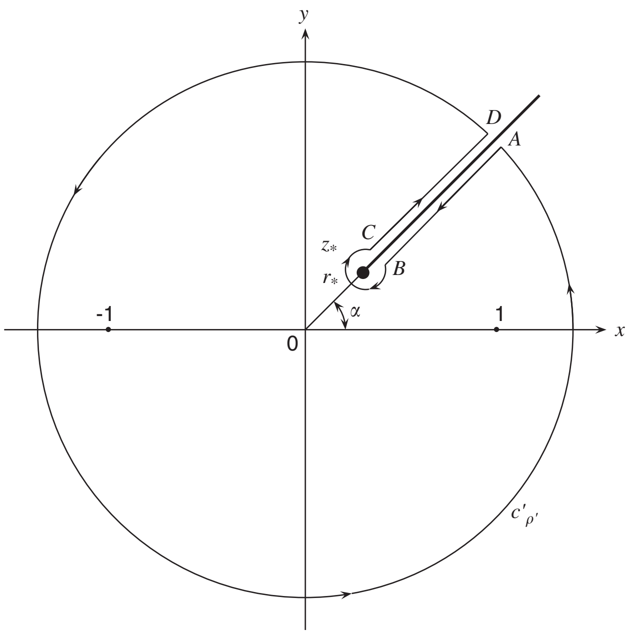

Fig. 1.

where P_n denotes the Legendre polynomial of degree n and Q_n is the Legendre function of the second kind defined, for $z \in \mathbb{C}\backslash[-1, 1]$, by

$$Q_n(z) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)\,\mathrm{d}t}{z-t}. \tag{4.4}$$

The function k_n is a fairly intractable one to do much with and it is useful to replace it by an asymptotic approximation which is valid for large n. From Donaldson and Elliott [3] we have that for $n \gg 1$ and $z \in \mathbb{C}\backslash[-1, 1]$

$$k_n(z) = \frac{c_n}{\left(z+\sqrt{z^2-1}\right)^{2n+1}}, \tag{4.5}$$

where we choose that branch of $\sqrt{z^2-1}$ such that $|z+\sqrt{z^2-1}| > 1$ and

$$c_n := \frac{2\pi(\Gamma(n+1))^2}{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}. \tag{4.6}$$

Although for n very large we see that c_n is approximately 2π we shall, in our numerical examples, leave c_n as defined by (4.6). To evaluate $E_n f$, we let the contour $\mathscr{C}_\rho$ go to infinity so that it becomes as illustrated in Fig. 1. The function $k_n f$ remains analytic within this contour. However, we are able to exploit the singularities of f and/or the integrals along the branch cuts in order to evaluate $E_n f$. We shall describe this in future by saying simply that we "let $\rho \to \infty$".

Let us now consider the integrals $I_k(g;\, j)$ for $k = 1, 2$ and 3 as defined in (1.1)–(1.6). From (1.1) and (4.2) we have

$$E_n I_k(g;\, j) = \frac{1}{2\pi\mathrm{i}}\int_{\mathscr{C}_\rho} k_n(z)g(z)j(z)h_k(z)\,\mathrm{d}z, \tag{4.7}$$

where initially $\rho(>1)$ is chosen so that $j(z)h_k(z)$ is analytic on and within $\mathscr{C}_\rho$ (recall that g is a polynomial). Recall also that we are assuming that the singularities of the Jacobian function j at $z_1, \overline{z_1}$ are further away from the interval $[-1, 1]$ than are the singularities $z_0, \overline{z_0}$ of the functions h_k, $k = 1, 2$ and 3.

Returning to (4.7), if z_* denotes a singular point of the integrand $k_n g j h_k$, we shall write $E_n I_k(g;\, j)(z_*)$ to denote the contribution to the truncation error $E_n I_k(g;\, j)$ from the neighbourhood of that singularity.

Let us illustrate these ideas with the simplest example which corresponds to $k=2$. From (1.3) we see that h_2 has a simple pole at z_0 with residue given by $1/(z_0-\overline{z_0})$. On letting $\rho \to \infty$ in (4.7) we find that

$$E_n I_2(g;\, j)(z_0) = -\frac{g(z_0)\,j(z_0)k_n(z_0)}{z_0-\overline{z_0}}. \tag{4.8}$$

Since the contribution to $E_n I_2(g;\, j)$ from the pole at $\overline{z_0}$ will be the complex conjugate of this we find, on using the asymptotic estimate for k_n as given in (4.5) and (4.6), that

$$\begin{aligned} E_n I_2(g;\, j)(z_0 \cup \overline{z_0}) &= E_n I_2(g;\, j)(z_0) + E_n I_2(g;\, j)(\overline{z_0}) \\ &= -2c_n \Re\left\{ \frac{g(z_0)\,j(z_0)}{(z_0-\overline{z_0})\left(z_0+\sqrt{z_0^2-1}\right)^{2n+1}} \right\}, \end{aligned} \tag{4.9}$$

for $n \gg 1$.

If the Jacobian function j is a constant then (4.9) gives the required asymptotic estimate of the truncation error. However, if j is of the form given by (1.6) then we must add to (4.9) the contribution to the truncation error from the algebraic singularities at the points z_1 and $\overline{z_1}$. From (4.7) we approximate to $E_n I_k(g;\, j)(z_1)$ by

$$E_n I_k(g;\, j)(z_1) = \frac{g(z_1)(z_1-\overline{z_1})^{1/2} h_k(z_1)}{2\pi \mathrm{i}} \int_{\mathscr{C}_\rho} k_n(z)(z-z_1)^{1/2}\,\mathrm{d}z. \tag{4.10}$$

From the discussion in the Appendix, and from (A.2) in particular, we find

$$\begin{aligned} E_n I_k(g;\, j)(z_1) &= g(z_1)(z_1-\overline{z_1})^{1/2} h_k(z_1) K_n(z_1;\, \tfrac{1}{2}) \\ &= \frac{c_n \mathrm{e}^{\mathrm{i}\pi/2}(z_1-\overline{z_1})^{1/2} g(z_1) h_k(z_1)(z_1^2-1)^{3/4}}{2\sqrt{\pi}(2n+1)^{3/2}\left(z_1+\sqrt{z_1^2-1}\right)^{2n+1}}, \end{aligned} \tag{4.11}$$

for $n \gg 1$. Since the contribution to $E_n I_k(g;\, j)$ from the branch point at $\overline{z_1}$ will be the complex conjugate of this we shall have for $n \gg 1$ and $k = 1, 2, 3$ that

$$E_n I_k(g;\, j)(z_1 \cup \overline{z_1}) = \frac{c_n}{\sqrt{\pi}(2n+1)^{3/2}} \Re\left\{ \frac{\mathrm{e}^{\mathrm{i}\pi/2}(z_1-\overline{z_1})^{1/2} g(z_1) h_k(z_1)(z_1^2-1)^{3/4}}{\left(z_1+\sqrt{z_1^2-1}\right)^{2n+1}} \right\}. \tag{4.12}$$

To sum up so far. For the integral $I_2(g;\, j)$, the truncation error for n-point Gauss–Legendre quadrature will be given by (4.9) if the Jacobian function j is a constant. However, if $j(z)$ is given by $\sqrt{(z-c)^2+d^2}$, see (1.6), then to (4.9) we must add (4.12) with $k=2$ and $h_2(z)$ given by (1.3) to obtain the truncation error estimate of

$$\begin{aligned} E_n I_2(g;\, j)(z_0 \cup \overline{z_0} \cup z_1 \cup \overline{z_1}) = -\,c_n \Re\Bigg\{ & \frac{2g(z_0)\,j(z_0)}{(z_0-\overline{z_0})\left(z_0+\sqrt{z_0^2-1}\right)^{2n+1}} \\ & - \frac{\mathrm{e}^{\mathrm{i}\pi/2}(z_1-\overline{z_1})^{1/2} g(z_1)(z_1^2-1)^{3/4}}{\sqrt{\pi}(2n+1)^{3/2}(z_1-z_0)(z_1-\overline{z_0})\left(z_1+\sqrt{z_1^2-1}\right)^{2n+1}} \Bigg\} \end{aligned} \tag{4.13}$$

for $n \gg 1$.

Consider now the integral $I_1(g;\,j)$ where h_1 is defined by (1.2). We see that h_1 has branch points at z_0, $\overline{z_0}$ and is given by $h_1(z)=\frac{1}{2}\log(z-z_0)+\frac{1}{2}\log(z-\overline{z_0})$. Let us first consider the contribution to $E_nI_1(g;\,j)$ from the branch point at z_0. We have approximately that

$$E_nI_1(g;\,j)(z_0)=\frac{1}{2}g(z_0)j(z_0)\left\{\frac{1}{2\pi\mathrm{i}}\int_{\mathscr{C}_\rho}k_n(z)\log(z-z_0)\,\mathrm{d}z+\log(z_0-\overline{z_0})\frac{1}{2\pi\mathrm{i}}\int_{\mathscr{C}_\rho}k_n(z)\,\mathrm{d}z\right\}. \tag{4.14}$$

From (A.1) and (A.3) we may rewrite this as

$$E_nI_1(g;\,j)(z_0)=\tfrac{1}{2}g(z_0)j(z_0)\{L_n(z_0;\,0)+\log(z_0-\overline{z_0})K_n(0)\}. \tag{4.15}$$

Since $K_n(0)=0$ we find from (A.3) that for $n\gg 1$

$$E_nI_1(g;\,j)(z_0)=-\frac{c_ng(z_0)j(z_0)(z_0^2-1)^{1/2}}{2(2n+1)\left(z_0+\sqrt{z_0^2-1}\right)^{2n+1}}. \tag{4.16}$$

Again, since the contribution from the branch point at $\overline{z_0}$ will be the complex conjugate of this we have

$$E_nI_1(g;\,j)(z_0\cup\overline{z_0})=-\frac{c_n}{2n+1}\Re\left\{\frac{g(z_0)j(z_0)(z_0^2-1)^{1/2}}{\left(z_0+\sqrt{z_0^2-1}\right)^{2n+1}}\right\} \tag{4.17}$$

for $n\gg 1$. If j is a constant then this is the required estimate for $E_nI_1(g;\,j)$. If, however, j is given by (1.6) then to (4.17) we must add the contribution to the error as given by (4.12) with $k=1$, and $h_1(z)$ given by (1.2), to give the overall truncation error estimate

$$\begin{aligned}E_nI_1(g;\,j)(z_0\cup\overline{z_0}\cup z_1\cup\overline{z_1})= &-\frac{c_n}{2n+1}\Re\left\{\frac{g(z_0)j(z_0)(z_0^2-1)^{1/2}}{\left(z_0+\sqrt{z_0^2-1}\right)^{2n+1}}\right.\\ &\left.+\frac{\mathrm{e}^{\mathrm{i}\pi/2}(z_1-\overline{z_1})^{1/2}g(z_1)\log((z_1-z_0)(z_1-\overline{z_0}))(z_1^2-1)^{3/4}}{2\sqrt{\pi}(2n+1)^{1/2}\left(z_1+\sqrt{z_1^2-1}\right)^{2n+1}}\right\}\end{aligned} \tag{4.18}$$

for $n\gg 1$.

Finally, we must consider the truncation error for the integral $I_3(g;\,j)$ where h_3 is defined in (1.4). Again the function h_3 has branch points at z_0, $\overline{z_0}$. In estimating the truncation error from the branch point at z_0 we have, from (1.4), (4.7) and (A.2), that approximately

$$E_nI_3(g;\,j)(z_0)=(z_0-\overline{z_0})^{\lambda}g(z_0)j(z_0)K_n(z_0;\,\lambda). \tag{4.19}$$

From (A.2) we have that for $n\gg 1$

$$E_nI_3(g;\,j)(z_0)=\frac{c_n\mathrm{e}^{-\mathrm{i}\pi\lambda}(z_0-\overline{z_0})^{\lambda}g(z_0)j(z_0)(z_0^2-1)^{(\lambda+1)/2}}{\Gamma(-\lambda)(2n+1)^{\lambda+1}\left(z_0+\sqrt{z_0^2-1}\right)^{2n+1}}. \tag{4.20}$$

Table 2
Truncation error $E_n I_k(g;\, j)$ for $n \gg 1$

	$k=1$	$k=2$	$k=3$
j a constant	(4.17)	(4.9)	(4.21)
j given by (1.6)	(4.18)	(4.13)	(4.22)

Again, since $E_n I_3(g;\, j)(\overline{z_0})$ will be the complex conjugate of this we have

$$E_n I_3(g;\, j)(z_0 \cup \overline{z_0}) = \frac{2c_n}{\Gamma(-\lambda)(2n+1)^{\lambda+1}} \Re \left\{ \frac{\mathrm{e}^{-\mathrm{i}\pi\lambda}(z_0 - \overline{z_0})^{\lambda} g(z_0) j(z_0)(z_0^2 - 1)^{(\lambda+1)/2}}{\left(z_0 + \sqrt{z_0^2 - 1}\right)^{2n+1}} \right\} \tag{4.21}$$

for $n \gg 1$. If j is a constant then this is the required estimate for $E_n I_3(g;\, j)$. If, however, j is given by (1.6) then to (4.21) we must add the contribution to the error as given by (4.12) with $k=3$, and $h_3(z)$ given by (1.4), to obtain the overall truncation error estimate

$$E_n I_3(g;\, j)(z_0 \cup \overline{z_0} \cup z_1 \cup \overline{z_1}) = \frac{c_n}{2n+1} \Re \left\{ \frac{2\mathrm{e}^{-\mathrm{i}\pi\lambda}(z_0 - \overline{z_0})^{\lambda} g(z_0) j(z_0)(z_0^2 - 1)^{(\lambda+1)/2}}{\Gamma(-\lambda)(2n+1)^{\lambda}\left(z_0 + \sqrt{z_0^2 - 1}\right)^{2n+1}} \right.$$
$$\left. + \frac{\mathrm{e}^{\mathrm{i}\pi/2}(z_1 - \overline{z_1})^{1/2} g(z_1)((z_1 - z_0)(z_1 - \overline{z_0}))^{\lambda}(z_1^2 - 1)^{3/4}}{\sqrt{\pi}(2n+1)^{1/2}\left(z_1 + \sqrt{z_1^2 - 1}\right)^{2n+1}} \right\} \tag{4.22}$$

for $n \gg 1$.

Before considering a similar analysis of the truncation errors when n-point Gauss–Legendre quadrature is applied to the transformed integrals, let us summarise the results of this section in Table 2.

5. Truncation error estimates for the transformed integrals

We now wish to estimate the truncation errors when n-point Gauss–Legendre quadrature is applied to the transformed integrals $I_k(g;\, j)$ as defined in Eqs. (3.4)–(3.9). If we write $w = u + \mathrm{i}v$ then we shall continue the definitions of the functions G, J and H_k, for $k = 1, 2$ and 3, into the complex w-plane. If we now denote the truncation error by $E_n^{\mathrm{tr}} I_k(g;\, j)$ we shall have as before, cf. (4.7),

$$E_n^{\mathrm{tr}} I_k(g;\, j) = \frac{1}{2\pi\mathrm{i}} \int_{\mathscr{C}_\rho} k_n(w) G(w) J(w) H_k(w)\, \mathrm{d}w. \tag{5.1}$$

The contour $\mathscr{C}_\rho$, for some $\rho > 1$, is chosen as one of the confocal ellipses in the w-plane, with foci at the points $(-1, 0)$ and $(1, 0)$, such that the function GJH_k is analytic on and within $\mathscr{C}_\rho$. As before, in order to obtain asymptotic estimates of the truncation error when n is large, we need to consider the singular points in the w-plane of the functions J and H_k. From the definitions of H_k in (3.7)–(3.9) we first need to determine points where $\cosh(\mu w - \eta) = 0$. There is an infinity of such points given by $w_k, \overline{w_k}$ say, where

$$w_k := \frac{\eta(a,b)}{\mu(a,b)} + \frac{(k+\frac{1}{2})\pi\mathrm{i}}{\mu(a,b)}, \quad k \in \mathbb{N}_0. \tag{5.2}$$

In order to estimate the truncation error we shall, in all cases, consider only the contributions from the singularities at $w_0, \overline{w_0}$ which are closest to the interval $[-1, 1]$. Recall that we have shown in Theorem 3.4 that the points $w_0, \overline{w_0}$

are further away from the interval $[-1, 1]$ than are the points $z_0, \overline{z_0}$. If $E_n^{\mathrm{tr}} I_k(g; j)(w_0)$ denotes the contribution to the truncation error from the singularity at w_0 then we shall, as a first step, write

$$E_n^{\mathrm{tr}} I_k(g; j)(w_0) = \frac{G(w_0)J(w_0)}{2\pi \mathrm{i}} \int_{\mathscr{C}_\rho} k_n(w)H_k(w)\,\mathrm{d}w, \tag{5.3}$$

approximately. From (3.5), (3.6), the definition of w_0 and recalling that $z_0 = a + \mathrm{i}b$ we have

$$E_n^{\mathrm{tr}} I_k(g; j)(w_0) = \frac{\mu g(z_0) j(z_0)}{2\pi \mathrm{i}} \int_{\mathscr{C}_\rho} k_n(w)H_k(w)\,\mathrm{d}w. \tag{5.4}$$

Let us evaluate this integral in the simplest case, when $k = 2$. Then H_2 has a simple pole at w_0 and we find from (3.8) that its residue is $2/(\mu(z_0 - \overline{z_0}))$. On letting $\rho \to \infty$ in (5.4) we find

$$E_n^{\mathrm{tr}} I_2(g; j)(w_0) = -\frac{2g(z_0)j(z_0)k_n(w_0)}{(z_0 - \overline{z_0})}. \tag{5.5}$$

Since $E_n^{\mathrm{tr}} I_2(g; j)(\overline{w_0})$ will be the complex conjugate of this we find, on using (4.5), that

$$E_n^{\mathrm{tr}} I_2(g; j)(w_0 \cup \overline{w_0}) = -4c_n \Re \left\{ \frac{g(z_0)j(z_0)}{(z_0 - \overline{z_0})\left(w_0 + \sqrt{w_0^2 - 1}\right)^{2n+1}} \right\} \tag{5.6}$$

for $n \gg 1$. If j is a constant then this is the required asymptotic estimate for $E_n I_2(g; j)$. On comparing (5.6) with (4.9) we see explicitly for the first time the advantage of the transformation. In the untransformed case, we see from (4.9) that the error tends to zero like $\mathrm{O}(1/|z_0 + \sqrt{z_0^2 - 1}|^{2n+1})$ whereas from (5.6) the transformed error tends to zero like $\mathrm{O}(1/|w_0 + \sqrt{w_0^2 - 1}|^{2n+1})$. From the discussion in Section 3 we know that $|w_0 + \sqrt{w_0^2 - 1}| > |z_0 + \sqrt{z_0^2 - 1}|$ so that the rate of convergence to zero in the transformed case will be better that in the untransformed case. We shall show some numerical results for specific examples in Section 6.

Suppose now that j is given by (1.6) so that from (3.6) we have

$$J(w) = (a + b\sinh(\mu w - \eta) - z_1)^{1/2}(a + b\sinh(\mu w - \eta) - \overline{z_1})^{1/2}. \tag{5.7}$$

This function will have branch points where $a + b\sinh(\mu w - \eta) = z_1$ and $a + b\sinh(\mu w - \eta) = \overline{z_1}$. Again there will be an infinity of such points in the w-plane. Let w_1^* and $\overline{w_1}^*$ be the points closest to $[-1, 1]$ such that

$$a + b\sinh(\mu w_1^* - \eta) = z_1; \tag{5.8}$$

that is

$$w_1^* = \frac{\eta}{\mu} + \frac{1}{\mu}\operatorname{arcsinh}\left(\frac{z_1 - a}{b}\right). \tag{5.9}$$

Let us consider $E_n^{\mathrm{tr}} I_k(g; j)(w_1^*)$ where k may be 1, 2 or 3. From (5.1) we have, first of all, that this is given approximately by

$$E_n^{\mathrm{tr}} I_k(g; j)(w_1^*) = \frac{G(w_1^*)H_k(w_1^*)}{2\pi \mathrm{i}} \int_{\mathscr{C}_\rho} k_n(w)J(w)\,\mathrm{d}w. \tag{5.10}$$

Now, near w_1^*, we have from (5.7) that

$$J(w) = (z_1 - \overline{z_1})^{1/2}(b(\sinh(\mu w - \eta) - \sinh(\mu w_1^* - \eta)))^{1/2} \tag{5.11}$$

approximately. On approximating $(\sinh(\mu w - \eta) - \sinh(\mu w_1^* - \eta))$ by $\mu(w - w_1^*)\cosh(\mu w_1^* - \eta)$ and observing from (5.8) that $b\cosh(\mu w_1^* - \eta) = ((z_1 - z_0)(z_1 - \overline{z_0}))^{1/2}$ we find that

$$J(w) = \mu^{1/2}(z_1 - \overline{z_1})^{1/2}((z_1 - z_0)(z_1 - \overline{z_0}))^{1/4}(w - w_1^*)^{1/2} \tag{5.12}$$

approximately, in the neighbourhood of w_1^*. Since $G(w_1^*) = \mu g(z_1)$ we find from (5.10), (5.12) and (A.2) that

$$E_n^{\rm tr} I_k(g;\, j)(w_1^*) = \mu^{3/2} g(z_1)(z_1 - \overline{z_1})^{1/2}((z_1 - z_0)(z_1 - \overline{z_0}))^{1/4} H_k(w_1^*) K_n(w_1^*;\, \tfrac{1}{2}) \tag{5.13}$$

for $n \geqslant 1$. From (A.2) again we will obtain the required asymptotic estimate for $E_n^{\rm tr} I_k(g;\, j)(w_1^*)$. Since the contribution to $E_n^{\rm tr} I_k(g;\, j)$ from the singularity at $\overline{w_1}^*$ will be the complex conjugate of this we find on addition that

$$E_n^{\rm tr} I_k(g;\, j)(w_1^* \cup \overline{w_1}^*) = \frac{c_n \mu^{3/2}}{\pi^{1/2}(2n+1)^{3/2}} \times \Re \left\{ \frac{{\rm e}^{{\rm i}\pi/2} g(z_1)(z_1 - \overline{z_1})^{1/2}((z_1 - z_0)(z_1 - \overline{z_0}))^{1/4}(w_1^{*2} - 1)^{3/4} H_k(w_1^*)}{\left(w_1^* + \sqrt{w_1^{*2} - 1}\right)^{2n+1}} \right\} \tag{5.14}$$

for $n \geqslant 1$. We note here that from (3.7)–(3.9) and (5.8) we have

$$H_1(w_1^*) = \tfrac{1}{2}((z_1 - z_0)(z_1 - \overline{z_0}))^{1/2} \log((z_1 - z_0)(z_1 - \overline{z_0})), \tag{5.15}$$

$$H_2(w_1^*) = ((z_1 - z_0)(z_1 - \overline{z_0}))^{-1/2}, \tag{5.16}$$

$$H_3(w_1^*) = ((z_1 - z_0)(z_1 - \overline{z_0}))^{\lambda + 1/2}. \tag{5.17}$$

On recalling (5.6), if j is given by (1.6) then from (5.14) and (5.16) we find

$$E_n^{\rm tr} I_2(g;\, j)(w_0 \cup \overline{w_0} \cup w_1^* \cup \overline{w_1}^*) = c_n \Re \left\{ \frac{{\rm e}^{{\rm i}\pi/2} \mu^{3/2} g(z_1)(z_1 - \overline{z_1})^{1/2}((z_1 - z_0)(z_1 - \overline{z_0}))^{-1/4}(w_1^{*2} - 1)^{3/4}}{\sqrt{\pi}(2n+1)^{3/2}\left(w_1^* + \sqrt{w_1^{*2} - 1}\right)^{2n+1}} - \frac{4 g(z_0) j(z_0)}{(z_0 - \overline{z_0})\left(w_0 + \sqrt{w_0^2 - 1}\right)^{2n+1}} \right\}. \tag{5.18}$$

We must now consider the estimates for $E_n^{\rm tr} I_k(g;\, j)(w_0 \cup \overline{w_0})$ when $k = 1$ and 3; recall that the estimate for $k = 2$ has been given in (5.6). From (5.4) we have

$$E_n^{\rm tr} I_1(g;\, j)(w_0) = \frac{\mu g(z_0) j(z_0)}{2\pi {\rm i}} \int_{\mathscr{C}_\rho} k_n(w) H_1(w)\, {\rm d}w \tag{5.19}$$

approximately, where $H_1(w)$ is defined by (3.7). We are interested in $H_1(w)$ near w_0. Since $\cosh(\mu w_0 - \eta) = 0$ and $\sinh(\mu w_0 - \eta) = {\rm e}^{{\rm i}\pi/2}$ it follows that near w_0

$$\begin{aligned} b \cosh(\mu w - \eta) &= b \cosh((\mu w_0 - \eta) + \mu(w - w_0)) \\ &= {\rm e}^{{\rm i}\pi/2} b \sinh(\mu(w - w_0)) \\ &= {\rm e}^{{\rm i}\pi/2} b\mu(w - w_0) \end{aligned} \tag{5.20}$$

approximately.

On substituting (5.20) into (3.7) we have that for w close to w_0

$$H_1(w) = {\rm e}^{{\rm i}\pi/2} b\mu[\log(b\mu {\rm e}^{{\rm i}\pi/2})(w - w_0) + (w - w_0)\log(w - w_0)]. \tag{5.21}$$

Substituting this for $H_1(w)$ into (5.19) and recalling the definitions of $K_n(m)$ and $L_n(z_*;\, m)$ from (A.1) and (A.3), respectively, we find

$$E_n^{\rm tr} I_1(g;\, j)(w_0) = {\rm e}^{{\rm i}\pi/2} b\mu^2 g(z_0) j(z_0)\{\log(b\mu {\rm e}^{{\rm i}\pi/2})[K_n(1) - w_0 K_n(0)] + L_n(w_0;\, 1)\}. \tag{5.22}$$

For all $n \in \mathbb{N}$ we know that $K_n(0) = K_n(1) = 0$ so that from (A.3) we find for $n \gg 1$ that

$$E_n^{\mathrm{tr}} I_1(g;\, j)(w_0) = -\frac{c_n \mu^2}{2(2n+1)^2} \frac{(z_0 - \overline{z_0}) g(z_0) j(z_0)(w_0^2 - 1)}{\left(w_0 + \sqrt{w_0^2 - 1}\right)^{2n+1}}, \tag{5.23}$$

on recalling (1.5). Since the contribution from the singularity at $\overline{w_0}$ will be the complex conjugate of this we find

$$E_n^{\mathrm{tr}} I_1(g;\, j)(w_0 \cup \overline{w_0}) = -\frac{c_n \mu^2}{(2n+1)^2} \Re \left\{ \frac{(z_0 - \overline{z_0}) g(z_0) j(z_0)(w_0^2 - 1)}{\left(w_0 + \sqrt{w_0^2 - 1}\right)^{2n+1}} \right\} \tag{5.24}$$

for $n \gg 1$. When the Jacobian j is a constant this is the estimate for the truncation error in this case. When j is given by (1.6) we must add to this the contribution from the singularities at w_1^* and $\overline{w_1}^*$ as given by (5.14) together with (5.15); this gives

$$\begin{aligned} &E_n^{\mathrm{tr}} I_1(g;\, j)(w_0 \cup \overline{w_0} \cup w_1^* \cup \overline{w_1}^*) \\ &\quad = \frac{c_n \mu^{3/2}}{(2n+1)^{3/2}} \Re \left\{ \frac{\mathrm{e}^{\mathrm{i}\pi/2} g(z_1)(z_1 - \overline{z_1})^{1/2}((z_1 - z_0)(z_1 - \overline{z_0}))^{3/4}(w_1^{*2} - 1)^{3/4} \log((z_1 - z_0)(z_1 - \overline{z_0}))}{2\sqrt{\pi}\left(w_1^* + \sqrt{w_1^{*2} - 1}\right)^{2n+1}} \right. \\ &\qquad \left. - \frac{\sqrt{\mu}(z_0 - \overline{z_0}) g(z_0) j(z_0)(w_0^2 - 1)}{\sqrt{2n+1}\left(w_0 + \sqrt{w_0^2 - 1}\right)^{2n+1}} \right\}. \end{aligned} \tag{5.25}$$

It remains to consider the estimate for $E_n^{\mathrm{tr}} I_3(g;\, j)(w_0 \cup \overline{w_0})$. From (3.9) and (5.4) we have

$$E_n^{\mathrm{tr}} I_3(g;\, j)(w_0) = \frac{\mu g(z_0) j(z_0)}{2\pi \mathrm{i}} \int_{\mathscr{C}_\rho} k_n(w)(b \cosh(\mu w - \eta))^{2\lambda+1}\, \mathrm{d}w \tag{5.26}$$

approximately. If $2\lambda \in \mathbb{N}_0$ we see that the integrand is an entire function on $\mathbb{C}\backslash[-1, 1]$. The integral may estimated by the method of steepest descents, but we shall not pursue this any further in this paper. Henceforth, we shall assume that 2λ is not an integer. In this case the integrand has branch points at w_0 and $\overline{w_0}$. On using (5.20) to approximate $b \cosh(\mu w - \eta)$ near w_0 and recalling (A.2) we find from (5.26) that

$$E_n^{\mathrm{tr}} I_3(g;\, j)(w_0) = b^{2\lambda+1} \mu^{2\lambda+2} g(z_0) j(z_0) \mathrm{e}^{\mathrm{i}\pi(\lambda+1/2)} K_n(w_0;\, 2\lambda + 1). \tag{5.27}$$

From (A.2) we have an asymptotic estimate for $K_n(w_0;\, 2\lambda + 1)$ for $n \gg 1$. Noting as before that $E_n^{\mathrm{tr}} I_3(g;\, j)(\overline{w_0})$ will be the complex conjugate of (5.27) we find on addition that

$$\begin{aligned} E_n^{\mathrm{tr}} I_3(g;\, j)(w_0 \cup \overline{w_0}) &= \frac{c_n \mu^{2\lambda+2}}{2^{2\lambda} \Gamma(-2\lambda - 1)(2n+1)^{2\lambda+2}} \\ &\quad \times \Re \left\{ \frac{\mathrm{e}^{-\mathrm{i}\pi(2\lambda+1)}(z_0 - \overline{z_0})^{2\lambda+1} g(z_0) j(z_0)(w_0^2 - 1)^{\lambda+1}}{\left(w_0 + \sqrt{w_0^2 - 1}\right)^{2n+1}} \right\} \end{aligned} \tag{5.28}$$

for $n \gg 1$ and provided 2λ is not an integer. When j is a constant then (5.28) gives the estimate of the truncation error. When the Jacobian j satisfies (1.6) we must add to this the contribution from the singularities at w_1^* and $\overline{w_1}^*$ as given

Table 3
Truncation error $E_n^{\rm tr} I_k(g;\, j)$ for $n \geqslant 1$

	$k=1$	$k=2$	$k=3$
j a constant	(5.24)	(5.6)	(5.28)
j given by (1.6)	(5.25)	(5.18)	(5.29)

in (5.14) where $H_3(w_1^*)$ is given by (5.17). We find

$$\begin{aligned} &E_n^{\rm tr} I_3(g;\, j)(w_0 \cup \overline{w_0} \cup w_1^* \cup \overline{w_1}^*) \\ &\quad = \frac{c_n \mu^{3/2}}{(2n+1)^{3/2}} \Re \left\{ \frac{{\rm e}^{{\rm i}\pi/2} g(z_1)(z_1-\overline{z_1})^{1/2}((z_1-z_0)(z_1-\overline{z_0}))^{\lambda+3/4}(w_1^{*2}-1)^{3/4}}{\sqrt{\pi}\left(w_1^*+\sqrt{w_1^{*2}-1}\right)^{2n+1}} \right. \\ &\qquad \left. + \frac{\mu^{2\lambda+1/2}{\rm e}^{-{\rm i}\pi(2\lambda+1)}(z_0-\overline{z_0})^{2\lambda+1} g(z_0) j(z_0)(w_0^2-1)^{\lambda+1}}{2^{2\lambda}\Gamma(-2\lambda-1)(2n+1)^{2\lambda+1/2}\left(w_0+\sqrt{w_0^2-1}\right)^{2n+1}} \right\}. \end{aligned} \tag{5.29}$$

We have summarised these results for the transformed integrals in Table 3.

6. Some numerical examples

Having obtained asymptotic estimates of the truncation errors in the quadrature formulae, let us now see how good they are. In Johnston and Elliott [11], it has been demonstrated at length how effective the transformation (1.7) can be. Here we shall consider just four examples which, in addition to demonstrating again the effectiveness of the transformation, also compares the asymptotic estimates with the actual truncation errors. Although these estimates can be excellent we shall also see that there may be limitations.

Example 1. Consider the integral I_1 where

$$I_1 := \frac{1}{2}\int_{-1}^{1} \frac{x}{2}(x-1)\log((x-a)^2+b^2)\,{\rm d}x. \tag{6.1}$$

With $a=\frac{1}{2}$ and $n=20$ we give, in Table 4, a comparison of the actual errors with the asymptotic estimates for b taking the values 0.1, 0.01 and 0.001.

We note firstly that, with $n=20$ in each case, the transformation essentially reduces the error by a factor of 10^{-6} and secondly, that the asymptotic estimates give at least one correct significant figure in comparison with the actual error.

Example 2. Consider the integral I_2 where

$$I_2 := \int_{-1}^{1} \frac{1-x^2}{x^2+b^2}\,{\rm d}x = -2 + \frac{2(1+b^2)}{b}\arctan\left(\frac{1}{b}\right), \tag{6.2}$$

with $b>0$. The integrand has simple poles at the points $\pm{\rm i}b$ and again let b take the values 0.1, 0.01 and 0.001. In the untransformed case let us choose $n=28$ while in the transformed case we shall let $n=10$. A comparison of the actual truncation errors with their asymptotic estimates for both cases is given in Table 5.

We first observe that the actual truncation errors in the transformed case with $n=10$ are substantially less than those in the untransformed case with $n=28$. This illustrates the effectiveness of the sinh transformation.

Secondly, we see that the asymptotic estimate of the error, using (4.9), is poor in the untransformed case for $b=0.01$ and 0.001 in that there are no correct significant digits. This suggests that the asymptotic estimate for k_n, as given by (4.5) and (4.6), is not good enough for such small values of b. In Elliott [4], alternative asymptotic estimates for k_n are

Table 4
Actual errors and asymptotic estimates for errors in the evaluation of the integral I_1 for various values of b for both the untransformed and transformed integrals with $n = 20$

b	Untransformed integral, $n = 20$		Transformed integral, $n = 20$	
	Actual error	Asymp. est. (4.17)	Actual error	Asymp. est. (5.24)
0.1	-1.6678×10^{-4}	-1.1371×10^{-4}	-6.4210×10^{-13}	-6.1832×10^{-13}
0.01	-9.7186×10^{-3}	-8.8025×10^{-3}	$+1.2757 \times 10^{-9}$	$+1.2498 \times 10^{-9}$
0.001	-1.1320×10^{-2}	-1.3531×10^{-2}	$+1.9104 \times 10^{-8}$	$+1.9278 \times 10^{-8}$

Table 5
Actual errors and asymptotic estimates for errors in the evaluation of the integral I_2 for various values of b for the untransformed integral with $n = 28$ and for the transformed integral with $n = 10$

b	Untransformed integral, $n = 28$		Transformed integral, $n = 10$	
	Actual error	Asymp. est. (4.9)	Actual error	Asymp. est. (5.6)
0.1	2.55×10^{-1}	2.12×10^{-1}	3.2802×10^{-3}	3.2386×10^{-3}
0.01	1.574×10^{2}	3.523×10^{2}	2.6894×10^{0}	2.6496×10^{0}
0.001	3.132×10^{3}	5.883×10^{3}	1.6581×10^{2}	1.6492×10^{2}

Table 6
Actual errors and asymptotic estimates for errors in the evaluation of the integral I_2 for various values of b for the untransformed integral with $n = 28$

b	Untransformed integral, $n = 28$	
	Actual error	Asymp. est. (6.4)
0.1	2.55×10^{-1}	2.14×10^{-1}
0.01	1.574×10^{2}	2.270×10^{2}
0.001	3.132×10^{3}	3.052×10^{3}

given which will be more appropriate in this case. From Eqs. (2.13) and (2.14) of [4] we find on taking the first terms only in these expansions that an alternative estimate for k_n when $n \gg 1$ is given by

$$k_n(z) = \frac{2K_0((n+1/2)\operatorname{arccosh} z)}{I_0((n+1/2)\operatorname{arccosh} z)}. \tag{6.3}$$

With this estimate for k_n we find (cf. (4.9))

$$E_n I_2 = \frac{2(1+b^2)}{b}\Re\left\{\frac{\mathrm{i}K_0((n+1/2)\operatorname{arccosh}(\mathrm{i}b))}{I_0((n+1/2)\operatorname{arccosh}(\mathrm{i}b))}\right\}. \tag{6.4}$$

In Table 6 we compare this asymptotic estimate with the actual errors when $n = 28$ and $b = 0.1$, 0.01 and 0.001. We see that the new asymptotic estimates are better than those given in Table 5 when $b = 0.01$ and 0.001. However, this comes at a cost of using modified Bessel functions rather than elementary functions.

Example 3. Consider the integral I_3 where

$$I_3 := \int_{-1}^{1} \frac{x}{2}(x+1)((x-a)^2+b^2)^{-0.4}\,\mathrm{d}x. \tag{6.5}$$

In this example we shall choose $a = 0.25$ and again let b take the values 0.1, 0.01, 0.001. The actual and asymptotic errors in the untransformed integral with $n = 25$ and in the transformed integral with $n = 15$ are displayed in Table 7.

Again, it is interesting to note that 15 point Gauss–Legendre quadrature applied to the transformed integral gives considerably smaller truncation errors than 25 point Gauss–Legendre quadrature in the untransformed case. Although most of the asymptotic estimates are good it should be noted that in the untransformed case when $b = 0.001$ the asymptotic estimate is about twice that of the actual error.

Table 7
Actual errors and asymptotic estimates for errors in the evaluation of the integral I_3 for various values of b for the untransformed integral with $n=25$ and for the transformed integral with $n=15$

b	Untransformed integral, $n=25$		Transformed integral, $n=15$	
	Actual error	Asymp. est. (4.21)	Actual error	Asymp. est. (5.28)
0.1	-6.5044×10^{-4}	-6.7175×10^{-4}	-2.1043×10^{-9}	-2.0584×10^{-9}
0.01	-2.6430×10^{-1}	-2.1443×10^{-1}	$+1.8532 \times 10^{-7}$	$+1.7603 \times 10^{-7}$
0.001	-4.8955×10^{-1}	-8.7740×10^{-1}	$+1.4016 \times 10^{-5}$	$+1.3770 \times 10^{-5}$

Table 8
Actual errors and asymptotic estimates for errors in the evaluation of the integral I_4 for various values of b for the untransformed integral with $n=30$ and for the transformed integral with $n=15$

b	Untransformed integral, $n=30$		Transformed integral, $n=15$	
	Actual error	Asymp. est. (4.13)	Actual error	Asymp. est. (5.18)
0.1	$+6.5008 \times 10^{-3}$	$+6.5187 \times 10^{-3}$	$+1.2636 \times 10^{-6}$	$+1.2501 \times 10^{-6}$
0.01	$+1.2485 \times 10^{2}$	$+3.3268 \times 10^{1}$	-6.7516×10^{-2}	-6.6706×10^{-2}
0.001	$+3.5953 \times 10^{3}$	$+7.3809 \times 10^{2}$	$+1.0042 \times 10^{1}$	$+9.9387 \times 10^{0}$

Table 9
Ratio of the actual and asymptotic error estimates for evaluation of the integral I_4 for the untransformed integral with $n=30$ and for the transformed integral with $n=15$

b	Untransformed integral, $n=30$	Transformed integral, $n=15$
	Actual/asymptotic	Actual/asymptotic
0.1	0.9970	1.0108
0.01	3.7530	1.0121
0.001	4.8710	1.0104

Example 4. In the final example we shall consider an integral where the Jacobian function j is not constant on the interval of integration. The Jacobian we have introduced arises when a curved quadratic boundary element is interpolated between the points $(1, 1)$, $(2, \frac{1}{2})$ and $(3, 1)$. Consider the integral

$$I_4 := \int_{-1}^{1} \frac{\sqrt{1+x^2}}{(x-a)^2+b^2}\,\mathrm{d}x. \tag{6.6}$$

On writing $z_0 = a + \mathrm{i}b$ it may be shown, on splitting the integrand into partial fractions, that

$$I_4 = 2\,\mathrm{arcsinh}\,1 - \frac{1}{b}\Im\left\{\sqrt{z_0^2+1}\log\left(\frac{\sqrt{2}z_0+\sqrt{z_0^2+1}}{\sqrt{2}z_0-\sqrt{z_0^2+1}}\right)\right\}. \tag{6.7}$$

In this example we shall choose $a = 0.75$ and let b take the values 0.1, 0.01 and 0.001. For the untransformed integral we shall choose $n = 30$ and, for the transformed integral, $n = 15$. The results are given in Table 8.

From Table 8 we note that 15 point Gauss–Legendre quadrature on the transformed integral gives a truncation error which is two or three orders of magnitude smaller than that for 30 point Gauss–Legendre quadrature on the untransformed integral. Although the asymptotic estimates compare well with the actual truncation errors in the transformed case, they are not as good in the untransformed case even though the value of n has doubled. In Table 9 we give the ratios of the actual to the asymptotic estimates arising from Table 8.

Since, as we have shown in Section 3, the transformation takes the singularities further away from the interval of integration $[-1, 1]$, it is perhaps not surprising that in the transformed case the estimates are within about 1% of the actual errors.

In conclusion, we have illustrated by considering four examples how, in many cases, the asymptotic estimate of the truncation error in the quadrature rule gives a good approximation to the actual error. However, it should be noted that the asymptotic estimates may not be good when the singularities of the integrand are very close to the interval of integration and n is not taken large enough. In all cases we have illustrated that the truncation error is reduced, at times dramatically, when Gauss–Legendre quadrature is applied to the transformed integral.

Appendix A. Evaluation of three contour integrals

It is convenient to consider here the evaluation of three contour integrals which arise in the analysis of Sections 4 and 5. Firstly, we will show that the function

$$K_n(m) := \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{C}_\rho} k_n(z) z^m \, \mathrm{d}z = 0, \tag{A.1}$$

where $m \in \mathbb{N}_0$ with $m \leqslant 2n-1$ and the contour $\mathscr{C}_\rho$ is any ellipse with $\rho > 1$. We will also show that

$$K_n(z_*; \lambda) := \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{C}_\rho} k_n(z)(z - z_*)^\lambda \, \mathrm{d}z = \frac{c_n \mathrm{e}^{-\mathrm{i}\pi\lambda}(z_*^2 - 1)^{(\lambda+1)/2}}{\Gamma(-\lambda)(2n+1)^{\lambda+1}\left(z_* + \sqrt{z_*^2 - 1}\right)^{2n+1}}, \tag{A.2}$$

and that

$$\begin{aligned} L_n(z_*; m) &:= \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{C}_\rho} k_n(z)(z - z_*)^m \log(z - z_*) \, \mathrm{d}z \\ &= -\frac{c_n m! (z_*^2 - 1)^{(m+1)/2}}{(2n+1)^{m+1}\left(z_* + \sqrt{z_*^2 - 1}\right)^{2n+1}} \end{aligned} \tag{A.3}$$

for $n \gg 1$. In these equations z_* is a point in the upper-half plane such that $z_* = r_* \mathrm{e}^{\mathrm{i}\alpha}$ where $r_* > 0$ and $0 < \alpha < \pi$; λ in (A.2) is not an integer and is such that $-1 < \lambda < 2n - 1$. Finally, in (A.3), $m \in \mathbb{N}_0$ with $0 \leqslant m \leqslant 2n - 1$. In both cases $\mathscr{C}_\rho$ is an ellipse with ρ chosen so that the point z_* is outside $\mathscr{C}_\rho$.

To begin, recall from (4.1) and (4.2) that $K_n(m)$ is simply the truncation error arising when n-point Gauss–Legendre quadrature is applied to the function x^m. It is well known, since we are assuming $m \leqslant 2n-1$, that the error is zero. We can also show this by observing first of all from (4.5), that if $z \in \mathscr{C}_\rho$ for any $\rho > 1$ then $|k_n(z)| \leqslant c_n/\rho^{2n+1}$, when $n \gg 1$. Since the length of the contour $\mathscr{C}_\rho$ is less that $\pi(\rho + 1/\rho)$, the circumference of the circle with centre at the origin and radius the semi-major axis of $\mathscr{C}_\rho$, and since $|z| \leqslant |z + \sqrt{z^2 - 1}| = \rho$ for $z \in \mathscr{C}_\rho$ it follows from (A.1) that

$$|K_n(m)| \leqslant c/\rho^{2n-m} \tag{A.4}$$

for some c independent of ρ. On letting $\rho \to \infty$ and recalling that $k_n(z)z^m$ is analytic for $z \in \mathbb{C}\backslash[-1, 1]$, we have that $|K_n(m)| = 0$ since we have assumed $m \leqslant 2n - 1$. Thus, $K_n(m) = 0$.

In order to evaluate K_n and L_n we shall cut the z-plane from $z_* = r_* \mathrm{e}^{\mathrm{i}\alpha}$ to $\infty \mathrm{e}^{\mathrm{i}\alpha}$ and then let $\rho \to \infty$. In order that the contour is deformed only over regions in the z-plane where the integrand is analytic, the contour $\mathscr{C}_\rho$ becomes $AB \cup CD \cup \mathscr{C}'_{\rho'}$, see Fig. 1. Along AB we have $z = z_* + r\mathrm{e}^{\mathrm{i}\alpha}$ with r from ∞ to 0; along CD, $z = z_* + r\mathrm{e}^{\mathrm{i}(\alpha - 2\pi)}$ with r from 0 to ∞. The contour $\mathscr{C}'_{\rho'}$ is that part of the ellipse $\mathscr{C}_{\rho'}$ from D to A, where ρ' is "large". From the conditions put on λ and m above we see that the integrals around $\mathscr{C}'_{\rho'}$ will tend to zero in both cases as $\rho' \to \infty$. Thus, we may rewrite $K_n(z_*;\lambda)$ and $L_n(z_*;m)$ as

$$K_n(z_*;\lambda) = -\frac{1}{\pi}\mathrm{e}^{-\mathrm{i}\pi\lambda}\sin(\pi\lambda)J_n(z_*;\lambda) \tag{A.5}$$

and

$$L_n(z_*;m) = -J_n(z_*;m), \tag{A.6}$$

respectively, where $J_n(z_*;\mu)$ for $-1\leqslant\mu\leqslant 2n-1$ is defined by

$$J_n(z_*;\mu) := \lim_{R\to\infty}\int_0^R k_n((r_*+r)\mathrm{e}^{\mathrm{i}\alpha})(r\mathrm{e}^{\mathrm{i}\alpha})^{\mu}\mathrm{e}^{\mathrm{i}\alpha}\,\mathrm{d}r, \tag{A.7}$$

where μ may be an integer. We shall estimate $J_n(z_*;\mu)$ for large n by using the asymptotic form of k_n as given by (4.5) and (4.6). We find

$$J_n(z_*;\mu) = c_n\lim_{R\to\infty}\int_0^R \frac{(r\mathrm{e}^{\mathrm{i}\alpha})^{\mu}\mathrm{e}^{\mathrm{i}\alpha}\,\mathrm{d}r}{((r_*+r)\mathrm{e}^{\mathrm{i}\alpha}+\sqrt{((r_*+r)\mathrm{e}^{\mathrm{i}\alpha})^2-1})^{2n+1}}. \tag{A.8}$$

To evaluate this integral, let us first write

$$(r_*+r)\mathrm{e}^{\mathrm{i}\alpha} = \cosh\theta, \quad r_*\mathrm{e}^{\mathrm{i}\alpha} = \cosh\theta_*, \quad \mathrm{e}^{\mathrm{i}\alpha}\,\mathrm{d}r = \sinh\theta\,\mathrm{d}\theta. \tag{A.9}$$

On recalling, see Abramowitz and Stegun [1, Section 4.6.32], that for $R\gg 1$, $\operatorname{arccosh}(R\mathrm{e}^{\mathrm{i}\alpha}) = \log(2R)+\mathrm{i}\alpha+\mathrm{O}(1/R^2)$ we find from (A.8) and (A.9) that

$$J_n(z_*;\mu) = c_n\lim_{R\to\infty}\int_{\theta_*}^{\log(2R)+\mathrm{i}\alpha}\mathrm{e}^{-(2n+1)\theta}\sinh\theta(\cosh\theta-\cosh\theta_*)^{\mu}\,\mathrm{d}\theta. \tag{A.10}$$

For $n\gg 1$, the major contribution to this integral will come from the neighbourhood of θ_*. On replacing $\sinh\theta$ by $\sinh\theta_*$ and $\cosh\theta-\cosh\theta_*$ by $(\theta-\theta_*)\sinh\theta_*$ we find

$$J_n(z_*;\mu) = c_n(\sinh\theta_*)^{\mu+1}\mathrm{e}^{-(2n+1)\theta_*}\lim_{R\to\infty}\int_{\theta_*}^{\log(2R)+\mathrm{i}\alpha}\mathrm{e}^{-(2n+1)(\theta-\theta_*)}(\theta-\theta_*)^{\mu}\,\mathrm{d}\theta. \tag{A.11}$$

Now from (A.9) we have $z_*=\cosh\theta_*$ so that $\sinh\theta_*=\sqrt{z_*^2-1}$ and $\mathrm{e}^{-\theta_*}=1/(z_*+\sqrt{z_*^2-1})$. On writing $\theta-\theta_*=\phi$, $\mathrm{d}\theta=\mathrm{d}\phi$ we have

$$\begin{aligned}\lim_{R\to\infty}\int_{\theta_*}^{\log(2R)+\mathrm{i}\alpha}\mathrm{e}^{-(2n+1)(\theta-\theta_*)}(\theta-\theta_*)^{\mu}\,\mathrm{d}\theta &= \lim_{R\to\infty}\int_0^{\log(2R)+\mathrm{i}\alpha-\operatorname{arccosh}(z_*)}\mathrm{e}^{-(2n+1)\phi}\phi^{\mu}\,\mathrm{d}\phi\\ &= \int_0^{\infty}\mathrm{e}^{-(2n+1)\phi}\phi^{\mu}\,\mathrm{d}\phi\end{aligned}$$

by Cauchy's theorem,

$$=\Gamma(\mu+1)/(2n+1)^{\mu+1}, \tag{A.12}$$

since $\mu>-1$; see, for example, Erdélyi [5, Section 4.3(1)]. From (A.11) and (A.12) we have for $n\gg 1$

$$J_n(z_*;\mu) = \frac{c_n\Gamma(\mu+1)(z_*^2-1)^{(\mu+1)/2}}{(2n+1)^{\mu+1}\left(z_*+\sqrt{z_*^2-1}\right)^{2n+1}}. \tag{A.13}$$

We can now give the required asymptotic estimates for the integrals. From (A.5) and (A.13) and the reflection formula for the gamma function [1, Section 6.1.17] we obtain (A.2) and from (A.6) and (A.13) we get (A.3).

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